Eur. Phys. J. B **25**, 391–396 (2002)
DOI: 10.1140/epjb/e20020043

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

A cellular automata model for social-learning processes in a classroom context

C.M. Bordogna[1,2] and E.V. Albano[1,a]

[1] Instituto de Investigaciones Fisicoquímicas Teóricas y Aplicadas, (INIFTA), CONICET, UNLP, Suc. 4, C.C. 16, (1900) La Plata, Argentina
[2] IMApEC, Facultad de Ingeniería, UNLP, La Plata, Argentina

Received 22 October 2001

Abstract. A model for teaching-learning processes that take place in the classroom is proposed and simulated numerically. Recent ideas taken from the fields of sociology, educational psychology, statistical physics and computational science are key ingredients of the model. Results of simulations are consistent with well-established empirical results obtained in classrooms by means of different evaluation tools. It is shown that students engaged in collaborative groupwork reach higher achievements than those attending traditional lectures only. However, in many cases, this difference is subtle and consequently very difficult to be detected using tests. The influence of the number of students forming the collaborative groups on the average knowledge achieved is also studied and discussed.

PACS. 87.23.Ge Dynamics of social systems – 01.40.Fk Physics education research (cognition, problem solving, etc.) – 02.70.-c Computational techniques – 89.90.+n Other topics in areas of applied and interdisciplinary physics

1 Introduction

During the last years the use of multidisciplinary research approaches for the study of complex processes and systems has gained growing acceptance in the scientific community [1]. Within this context, the advancement of social sciences also requires analytical and numerical approaches suitable to describe how social mechanisms can explicate different types of social dynamics behavior. Therefore, it is not surprising that the application of physical paradigms to achieve quantitative descriptions of social [2–11] and economical [12–16] processes has attracted a lot of interest.

The aim of this manuscript is to propose and study a model for social teaching-learning processes that take place in a classroom context. Early studies on learning processes have been conducted by psychologists and sociologists [17,18]. However, the topic is so difficult and complex that it has soon evolved into a field of multidisciplinary research [19]. The proposed model takes into account recent ideas on educational psychology suggesting that learning processes occur while people participate within social communities [19,20]. According to these concepts, individuals are active agents that participate in the evolution of the knowledge, in contrast to old-fashioned ideas suggesting that learning is merely the reception of factual knowledge or information.

Teaching-learning process that take place in a classroom context (TLC) are being extensively investigated [19]. These kinds of studies have the advantage that the knowledge of the students can be followed and evaluated. In particular, the study of the processes of learning and understanding physics and mathematics has additional advantages because these subjects are based on well defined conceptual frameworks [21]. Therefore, an increasing number of physicists and mathematicians have also been attracted to the study of cognitive processes and teaching-learning strategies [22].

In spite of considerable effort and the progress achieved in this field, a theory (mathematically tractable) capable to capture the main features of TLC remains to be developed. In this work, we propose a model for the TLC based on a multidisciplinary approach that links psychological and sociological theories of impact [23], educational psychology concepts [19] and well established methods and procedures of computer science and statistical physics [24].

The theory of social impact has been formulated by Latane [23] who claimed, based on empirical results, that the impact of a group of individuals on a given person depends at least of three factors: a) The "strength"of the numbers of the group as a measure of their credibility to persuade and become supported, b) their "immediacy" that accounts for the social "distance" among individuals and c) the number of individuals that plays a role in the so called "division of the impact". The concept of social impact can describe a wide variety of situations in which

[a] e-mail: ealbano@inifta.unlp.edu.ar

social impact is exerted, regardless of the form that such influence takes place, see *e.g.* [25] and references therein. After the statistical mechanic formulation of the theory of social impact [25] an increasing number of works, mostly devoted to the study of the dynamics of opinion formation and the emergency of leaders [5,26–28], have been published.

The manuscript is organized as follows: in Section 2 the model is presented and some concepts of the theory of social impact are adapted to teaching-learning processes. Also, details on the numerical simulation method are outlined. Results obtained by means of Monte Carlo simulations of the proposed model are presented and discussed in Section 3. Finally, we state our conclusions in Section 4.

2 Description of the model and the numerical method

The framework adopted for the formulation of the TLC model is similar to those used used to treat spin systems [24,29] and neural networks [30]. So, let us now define and discuss relevant concepts: The cognitive impact (CI) acting on an individual (or student) is the overall result of those interactions with his/her environment (teachers, peers, bibliography, etc.), capable of modifying his/her knowledge and the self-elaboration of such influence. The individual is an active transformer rather than a passive recipient of the CI. He/she can also become a source of CI to other individuals by persuading and supporting. The persuasiveness, $P_{ij} \geq 0$, describes the degree to which the ith individual can persuade the jth individual. Furthermore, during a discussion, the support, $S_{ij} \geq 0$, describes the degree to which the ith individual supports the statements of the jth individual. Within an interactive group, both S_{ij} and P_{ij} become enhanced when individuals share similar ideas about the subject under examination, they have social and cultural affinities (*i.e.* short "social distance"), etc. Persuasiveness and support are well established concepts in the field of quantitative socio-dynamics, so for further details see *e.g.* [2]. The knowledge of the jth individual, ($\sigma_j(t)$), at time t, is defined as a dynamic variable such as $-1 \leq \sigma_j(t) \leq 1$, where $\sigma_j(t) = 1$ corresponds to optimum knowledge.

Based on these considerations, we propose that the CI of the teacher on the jth student ($CI^{\rm TS}(j,t)$), can be written as

$$CI^{\rm TS}(j,t) = P_{{\rm T}j}(1 - \sigma_j(t)\sigma_{\rm T}), \quad (1)$$

where $\sigma_{\rm T} > 0$ and $P_{{\rm T}j}$ are the knowledge of the teacher and his/her ability to persuade the jth individual, respectively. $P_{{\rm T}j}$ depends on many factors, characteristic of both the teacher her/himself and the teacher-individual relationship, such as *e.g.* the rhetorical ability and the persuasive skills of the teacher, the didactic presentation of the subject of study, etc. Notice that $CI^{\rm TS}$ is minimal for $\sigma_j = 1$ and $\sigma_{\rm T} = 1$, because this situation corresponds to the impact between two individuals having the same (maximum) knowledge. Also, $CI^{\rm TS}$ is maximal for $\sigma_j = -1$ and $\sigma_{\rm T} = 1$, due to the largest difference in the knowledge. It should be admitted that, if student A knows almost nothing, it may be questionable the she/he benefits more from interacting with a knowledgeable teacher, than student B does, who knows already a little. This kind of effect could be introduced in the model, *e.g.* considering a non-lineal factor of the type $CI^{\rm TS}(j,t) = P_{{\rm T}j}(1 - \sigma_j(t)\sigma_{\rm T})^{\gamma}$, where γ is an exponent. However, at the present stage of the development of the model we have restricted ourselves the the simpler lineal dependence given by equation (1).

Within groups of N individuals, the CI of the student-student interaction (supervised by the teacher) $CI^{\rm SS}(j,t)$, is given by

$$CI^{\rm SS}(j,t) = \sum_{i=1,i\neq j}^{N} [P_{ij}(t)(1 - \sigma_i(t)\sigma_j(t)) - S_{ij}(t)(1 + \sigma_i(t)\sigma_j(t))]{\rm sign}(\sigma_i(t)/\sigma_{\rm T}), \quad (2)$$

where, within brackets, the first (second) term accounts for the mutual persuasiveness (support). The structure of these two terms is similar to that of equation (1) and it is plausible since it is expected that mutual support will be larger when the individuals have similar knowledge ($\sigma_i\sigma_j > 0$) while persuasiveness is expected to play a more relevant role in the opposite case ($\sigma_i\sigma_j < 0$). It is also assumed that both S_{ij} and P_{ij} are composed of intrinsic and extrinsic (or interactive) factors, so

$$S_{ij}(t) = S^0_{ij}(\sigma_{\rm T} + \sigma_i(t)), \quad (3)$$

and

$$P_{ij}(t) = P^0_{ij}(\sigma_{\rm T} + \sigma_i(t)), \quad (4)$$

where the intrinsic factors, S^0_{ij} and P^0_{ij}, depend on many causes such as the strength of psychological coupling, affinity of social status, education, rhetorical abilities, personal skills, etc. The extrinsic factor is provided by a comparison established by the individual with the teacher who assumes a leadership role. This factor is included to account for the fact that the model attempts to describe supervised collaborative group work [31]. In fact, the term $(\sigma_{\rm T} + \sigma_i(t))$ means that both persuasiveness and support between individuals could be either reinforced or weakened when the knowledge of the teacher is taken as a reference level. In addition, the term $B \equiv {\rm sign}(\sigma_i(t)/\sigma_{\rm T})$ in equation (2), explicitly accounts for the plausible fact that an individual with knowledge below the average ($\sigma_i < 0$) has low chance to cause an increment of the knowledge of another individual that is above the average ($\sigma_j > 0$). Also, due to this term, in the inverse case ($\sigma_i > 0, \sigma_j < 0$), the jth individual has great chance to increase his/her knowledge. It should be noticed that $CI^{\rm SS}$ may be either positive, negative or zero. These values, that at first glance may appear meaningless, will become clearly plausible after the formulation of equation (5), see also below.

The knowledge is a dynamic variable influenced by the CI. So, during a time interval Δt, the knowledge changes as follows: $\sigma_j(t + \Delta t) = \sigma_j(t) \pm \Delta\sigma$, where for the purpose of the calculation σ_j is assumed to be discrete so that $\Delta\sigma$ is the "quantum" of knowledge. Also, $\sigma_j(t)$ may improve (become worse) with the probability $P_j = \tau_j/(1 + \tau_j)$ and $(1 - P_j)$, where τ_j is a generalized Metropolis rate [24] given by

$$\tau_j = \exp^{\beta_{\rm TS} CI^{\rm TS}(j,t) + \beta_{\rm SS}(N) CI^{\rm SS}(j,t)}, \quad (5)$$

where each process has it own "noise" given by $1/\beta_{\rm TS}$ and $1/\beta_{\rm SS}(N)$, respectively. In fact, for the teaching-student relationship, the noise is due to misunderstandings, disorder in the classroom, inappropriate teaching material, lack of attention of the students, obscure explanations, etc. For the student-student interactions the noise $1/\beta_{\rm SS}(N)$ appears due to disordered discussions, misunderstandings, the lack of a well-organized participative activity, etc. In this case, the dependence on the number of students N has explicitly been considered to account for the division of the impact observed upon interactions within groups [23].

Recalling that the selected framework is similar to that used to treat spin systems [24,29,30], one may loosely use the terminology of statistical physics to associate the CI to the Hamiltonian and the knowledge to a spin variable. Furthermore, the operation of different noises ($\beta \equiv 1/kT$) in equation (5) corresponds to competitive dynamical processes, each of them with their own "social temperature". Also, the coupling constant is replaced by persuasiveness and support.

It should be noted that in order to fully understand the plausibility of equations (1–2), one has to analyze them in connection to equation (5). In fact, if $CI > 0$ the jth individual has a large chance to enhance his/her knowledge, as typically found for $\sigma_i > 0, \sigma_j < 0$. For $CI < 0$ one has the opposite situation as usually occurs for $\sigma_i < 0, \sigma_j > 0$.

The dynamic evolution of the system is simulated by means of a standard Monte Carlo technique [24]. During each Monte Carlo time step (mcs) the knowledge of all the students are update simultaneously as in the case of a cellular automata approach [32].

In order to perform the simulations it is assumed that $\sigma_{\rm T} = 1$, $P_{{\rm T}j} = 1 \forall j$, with $j = 1, ..., N_{\rm T}$, where $N_{\rm T}$ is the total number of students in the classroom. Also, S^0_{ji} and P^0_{ji} are assumed to be randomly distributed in the interval $(0, 1)$, so their average value over the whole classroom is close to $1/2$. The quantum of knowledge is taken as $\Delta\sigma = 0.1$, so the knowledge becomes a spin variable with 21 accessible states. The initial knowledge of the students is assumed to be uniformly distributed among these accessible states, so that $\langle\sigma\rangle = 0$.

3 Results and discussion

Figure 1 shows the time evolution of the knowledge of the students as obtained taking $N_{\rm T} = 96$ and $\beta_{\rm TS} = \beta_{\rm SS} = 6$. The curve (a) shown in Figure 1 corresponds to simulations performed assuming the teaching–student interaction only (Eq. (1)) and neglecting equation (2). This curve corresponds to the traditional teaching-learning method where the teacher is the primary source of knowledge and the role of the students is restricted to a passive reception of the information. Within the context of educational psychology this methodology corresponds to a behaviorist-derived theoretical perspective of the teaching-learning process [33]. It is the observed that the knowledge steadily increases during a transient period (say up to $t \approx 200$ mcs) and subsequently it reaches a saturation level. Such saturation gives the maximum knowledge ($\sigma_{\rm M}$) that can be achieved under the assumptions already discussed.

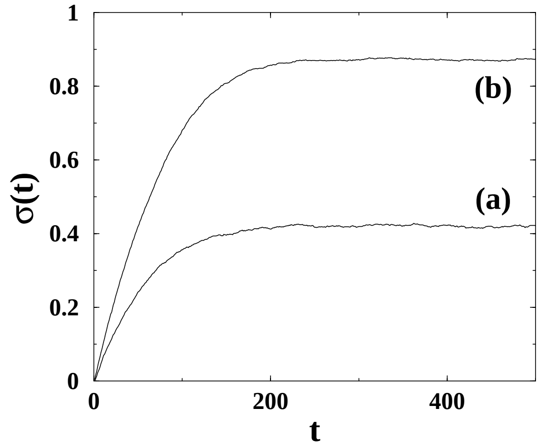

Fig. 1. Plots of the average knowledge $\sigma(t)$ *versus* time is obtained for (a) students attending the lecture of the teacher only (traditional approach) and (b) students engaged in collaborative group work. Results obtained assuming ($\beta = 6$), $N_{\rm T} = 96$ and averaging over 10^5 different samples.

The curve (b) in Figure 1 was obtained assuming that the students not only attend the lectures of the teacher (Eq. (1)) but also they are engaged in a cooperative groupwork (Eq. (2)). The groups are constituted by $N_{\rm G} = 3$ students and the individuals of each group are selected at random. This methodology corresponds to the modern approach of groupwork that within the context of educational psychology can be considered as a constructivist-derived theoretical perspective [33]. Curve (b) in Figure 1 also shows that at early times ($t \leq 200$ mcs) the knowledge of the students steadily increases and after a long time it reaches a saturation value. Comparing curves (a) and (b) of Figure 1 it becomes clearly evident that the achievements of the students engaged in collaborative group work are better than those attending the lectures of the teacher only.

Extensive test studies have demonstrated that in a large number of cases it is difficult to find differences between the traditional teaching method and the modern approach of groupwork [34]. In order to help to the understanding of these observations, simulations with groups in different environments have also been performed. For this purpose, the initial knowledge of the students is assumed to be uniformly distributed, $-1 \leq \sigma_j(t = 0) \leq 1$. Furthermore, groups of $N_{\rm G} = 3$ students are considered [35]. Figure 2 shows plots of the maximum knowledge achieved after a long instructional time ($\sigma_{\rm M}$) as a

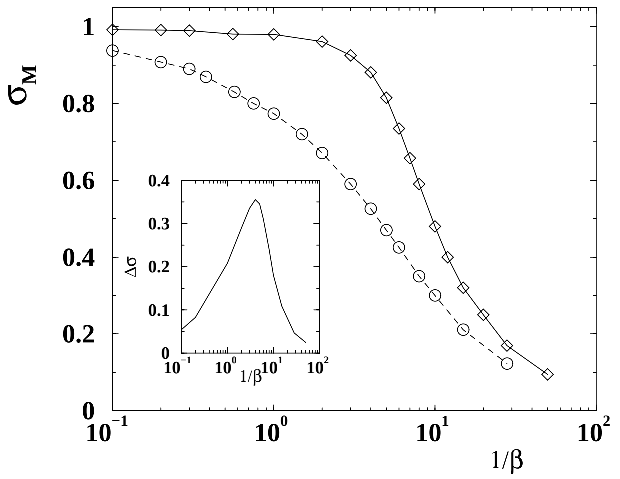

Fig. 2. Plots of the maximum achieved knowledge ($\sigma_{\rm M}$) *versus* the noise $1/\beta$, as obtained for: a) ◯, individuals only attending lectures of the teacher, b) ◊, individuals as in a) but also engaged in collaborative work forming groups of three members. The inset shows the difference between cases b) and a). The total number of individuals is $N_{\rm T} = 96$ and results are averaged over 10^5 different cases.

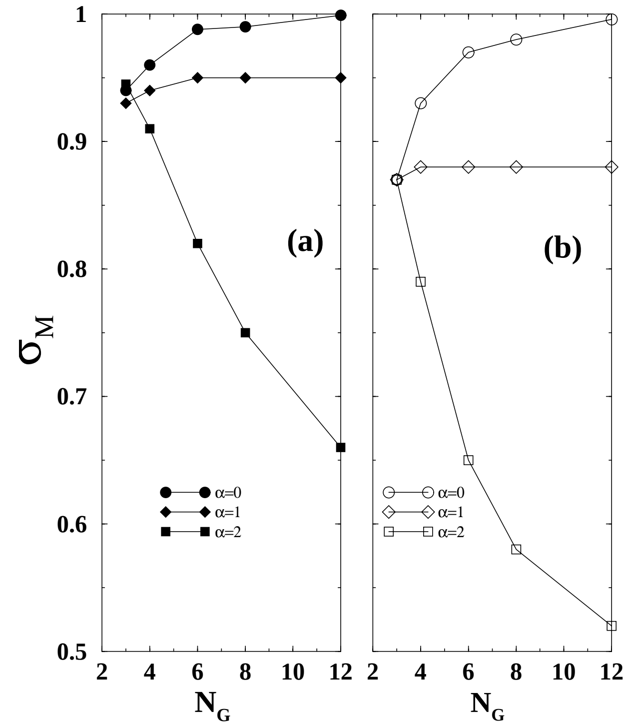

Fig. 3. Plots of the maximum achieved knowledge ($\sigma_{\rm M}$) *versus* the group size $N_{\rm G}$ obtained for different values of α as indicate in the figures (a) $1/\beta = 1/4$ and (b) $1/\beta = 1/6$. Results obtained for $N_{\rm T} = 96$ and averaged over 10^5 different samples.

function $1/\beta$, with $\beta_{\rm TS} = \beta_{\rm SS}$. It is found that using the traditional method $\sigma_{\rm M}$ decreases steadily when the noise is increased. In contrast, the knowledge achieved in collaborative groups is more robust and exhibits a sharp drop only for a larger noise close to $1/\beta \approx 6$. It is found that collaborative work always improves the achievements (see the inset of Fig. 2). However, the achievements of students attending lectures delivered by very good teachers (smaller values of $1/\beta$) can only slightly be improved by the groups, a fact that it makes difficult to detect differences using tests. This is also difficult in the other extreme case, *e.g.* for bad teachers and noisy groups. There is, of course, an intermediate regime where the difference becomes maximum and tests have great chances to detect the improvement caused by the collaborative work [34].

The plots of Figure 2 are similar to those used to locate phase transition in spin systems. In the present case the role of the order parameter (magnetization) is taken by the maximum achieved knowledge $\sigma_{\rm M}$, while the horizontal axis is given by $1/\beta \equiv KT$; *i.e.* the social temperature. So, the smooth charge of $\sigma_{\rm M}$ when T is increased observer for the traditional approach could roughly be interpreted as a second-order like critical behavior. Here the transition driven by the social temperature occurs between an "orderer state" that is characterized by excellent and very good achievements of the students and a "disordered state" where the achievements of the students are very poor. Considering the interaction among the students in collaborative groupwork the nature of the transition-like behavior changes into a first-order type, exhibiting an abrupt jump between the ordered and the disordered states. Of course, these "transitions" are heavily rounded due to the finite number of students involved in both the classroom and the groups. Well defined transitions may occur only in the "thermodynamic" limit $N_{\rm T} \to \infty$ and $N_{\rm G} \to \infty$, *i.e.* a quite unrealistic limit for the case of a TLC process.

The achievements of the students as a function of the number of members of the groups have also been studied, as it is shown in Figure 3. As follows from Figure 3 (topmost curves), one may considerably increase the achievements of the students simply increasing the size of groups toward unrealistic large amount of members. This figure dramatically point out that the division of the impact, as early proposed by Latané [23], has to be explicitly considered to model a TLC process.

In order to introduce the division of the impact, let us first point out that there is not an general agreement among different authors on the optimal group size. Such size may depend on the nature of the task as well as on the experience of the group members. Group sizes between two and six individuals have been recommended in various contexts, see *e.g.* [36,31,37,38]. There is a tendency to admit that the formation of pairs does not conform the "critical mass" to achieve significant learning. Some cooperative learning methods advocate using four-persons group. However, in this case the segregation of one individual (*e.g.* the most timid) usually has been observed. Larger groups make it possible for students to shirk responsibilities. In the present work $N_{\rm G} = 3$ is assumed as the optimal group size. However, it should be noted that this assumption does not affect our general conclusions and calculations with different values of $N_{\rm G}$ can straightforwardly be performed. So, in order to explicitly account

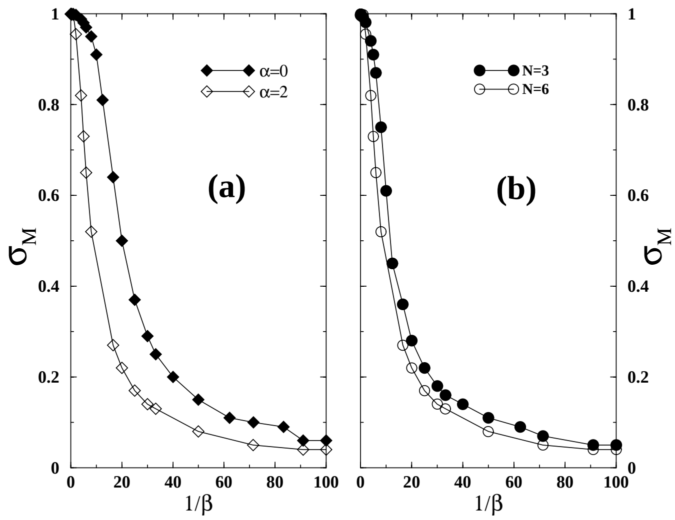

Fig. 4. Plots of the maximum achieved knowledge (σ_M) *versus* the noise $1/\beta$, obtained as follow: (a) keeping $N_G = 6$ constant and assuming two values of α as indicated in the figure and (b) keeping $\alpha = 2$ constants and assuming two values of N_G as indicated in the figure. Results obtained taking $N_T = 96$ and averaged over 10^5 different samples.

for the division of the impact it is assumed that the noise due to the student-student interaction depend on N_G according to

$$\beta_{SS} = \beta/(N_G/3)^\alpha, \tag{6}$$

where α is an exponent. Notice that for $\alpha = 0$ the division of the impact is ignored, as shown in the topmost curves of Figures 3a and b. Considering a rather weak division of the impact ($\alpha = 1$) the maximum knowledge achieved is almost independent of N_G (Fig. 3). However, for $\alpha = 2$ the division of the impact is relevant and the achievements decrease markedly upon increasing N_G, as shown in Figure 3. These results indicate that the division of the impact actually operates when exponents such as $\alpha > 1$ are considered in equation (5).

The interplay between noise and the division of the impact has also been analyzed as shown in Figure 4. Keeping the members of the group constant ($N_G = 6$) in Figure 4a, the typical "transition" like behavior is observed in plots of σ_M *versus* $1/\beta$. As expected, better achievements are always obtained when the division of the impact is ignored ($\alpha = 0$). While for small (large) noise such as $1/\beta \to 0$ ($1/\beta > 30$) the difference between the achieved knowledge with $\alpha = 0$ and $\alpha = 2$ is almost irrelevant, dramatic differences are observed within the intermediate regime, *i.e.* $5 \leq 1/\beta \leq 30$, as shown in Figure 4a.

Figure 4b shows plots of σ_M *versus* $1/\beta$ obtained keeping $\alpha = 2$ constant and taking two cases, namely $N_G = 3$ and $N_G = 6$, respectively. Since $N_G = 3$ is assumed as the optimum group size, the achievements are better for that case than for $N_G = 6$. However, the difference between those cases is rather small (Fig. 4b) as compared with the previously discussed example (Fig. 4a). So, assuming that the division of the impact may be a realistic approach to typical TLC processes, our finding suggest that a desirable pedagogical strategy may be focused to the reduction of the effects of the impact division prevailing in noisy group-work. It will certainly be very interesting to check these predictions of the model performing suitable test in classrooms.

4 Conclusion

A model for the theoretical description of social learning processes in a classroom context is proposed and studied numerally. The model predicts that the modern pedagogical approach of collaborative group allows the students to reach better achievement that in the case of traditional lectures. While the improvement is almost irrelevant when such traditional lectures are delivered by excellent teachers, in the typical cases (good and average teachers) the group work method makes a substantial difference. These findings are consistent with available results obtained testing the students, but additional tests will certainly be necessary (and welcomed!) in a order to check specifically the predictions of the model.

The model also suggest that pedagogical strategies may account for a drastic suppression of the noise caused by the division of the impact in order to allow better achievements when classroom constrains (*e.g.* sophisticated laboratory experiments) do not allow to keep the group size within optimal values.

It should be mentioned that the model can be extended in order to treat the influence of the structure of the collaborative groups on the achievement of the individuals [39], as well as to describe social-learning processes established through the Internet [40].

We hope that this preliminary theoretical approach for the social learning process will stimulate the development of improved socio-dynamic theories of learning aimed to help in the design of better pedagogic strategies.

Furthermore, the model may contribute to the exciting possibility that complex social human behavior may be studied with the aid of methods and concepts developed in the field of statistical physics.

This work was supported by UNLP, CONICET and ANPCyT (Argentina). The authors acknowledge the kind hospitality of Prof. K. Binder in the University of Mainz (Germany). C.B. acknowledges the support of the FOMEC project.

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